# Technical Change in a 3x2x2 Factor Endowment Model of Trade

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**Abstract**

The derivation of an algebraic factor endowment model of trade with three goods, two factors, two countries and technical change is demonstrated. An analysis of the comparative statics is performed in the form of a numeric example and leads to interesting results. The idea of "endogenous technical change" is presented as a way to improve the model.

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# 1 Introduction

Higher dimensions in traditional factor endowment trade theory are in general interesting as they extend long established and often-taught cornerstones of neoclassical trade theory. The issue of technological change and growth in a trading world has a long tradition in the literature. Nevertheless, technological change in higher dimensional trade models is not yet very well understood. One reason for this deficiency is the popularity of imperfect competition models in international economics during the last two decades.

In a higher dimensional trade model, the existence of different cones of diversification is not only possible, but also of utmost importance for the debate on trade and factor prices. In a 2x2 factor proportions model of trade, the statement that countries' endowments lie within one diversification cone is equivalent to the statement that countries' endowments are not too different to allow for factor price equalization (FPE).[1]

Higher dimensions in a factor proportions model give rise to the possibility of several cones of diversification. Figure 1 illustrates a case with three goods ($X$, $Y$, $Z$). Good prices are such that two cones of diversification exist. The position of the unit-value-isoquants depends on good prices and technology. In a situation such as shown in figure 1, there is no single factor-price- or isocost-line that is consistent with non-negative profits and cost-minimizing behavior of firms. Instead, two sets of factor prices exist ($w_A, r_A$ *vs.* $w_B, r_B$). With intersectoral mobility of factors, there cannot exist two sets of factor prices within one country. Thus, countries specialize - given internationally common good prices and technology - in their production.

The two corresponding cones of diversification are shown in figure 1. The pattern of specialization follows countries' endowments: Countries that are relatively well endowed with labor ($L$) specialize in the production of the relatively labor intensive goods and vice versa. Factor prices are not equalized in this situation but differ according to different "membership" to cone 1 or cone 2 (or to none of them). Countries within the same cone share factor prices. It is interesting to note that for a country that lies in – say – cone 1, the accumulation of capital will not immediately result in a change in its factor prices as in the Solow-Growth model for a closed economy. In a situation with several cones of diversification, trade isolates a country from the effects of factor accumulation on factor prices as long as it stays within the same cone.

A trade related influence of technological change on factor prices and the likelihood of FPE is now obvious. Any change in technology alters the position of the unit-value-isoquants in figure 1. This changes the position of the cones and

[1] Other necessary assumptions for the FPE-theorem to hold are that there are no factor-intensity reversals and that all economies are fully diversified (produce both goods).

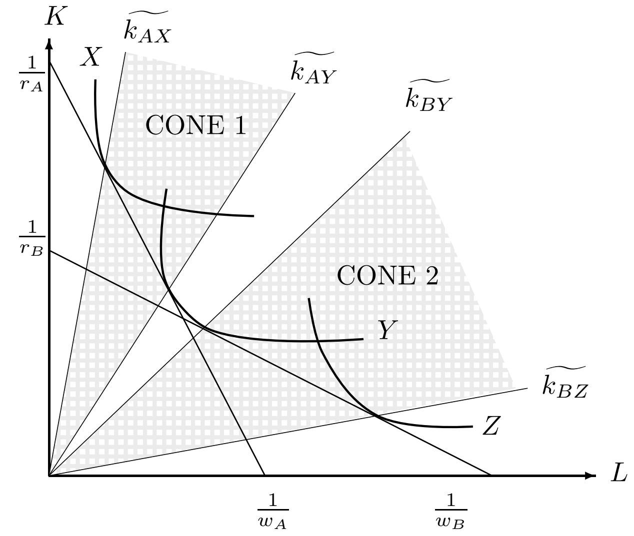


Figure 1: Two cones in a 3x2x2 trade model

that of the unit-cost- or factor-price-lines. In Becker / Kunz (2003), we assume Harrod-neutral technological progress in the capital-intensive sector and show that this encourages factor price equalization. We relate this to the neoclassical growth model and challenge the pessimistic view of Deardorff (2001) who argues that "there seems to be nothing in the available neoclassical growth models to suggest that the growth process is likely to bring the countries of the world sufficiently close together in terms of their factor endowments to permit global FPE" (Deardorff 2001: 197).

For this kind of graphical analysis, technological progress is introduced in a general equilibrium but price variations are ruled out in order to isolate the effect of technological change on the specialization pattern and on factor prices. To account for the endogeneity of prices, it is necessary to analyze the general equilibrium algebraically with techniques as promoted by Dixit / Norman (1980).

The paper proceeds as follows: Section 2 establishes the general equilibrium model for a 3x2x2 world with mobility in goods and immobile factors of production between two countries. The initial situation of the world economy is set out by assuming two cones of diversification as shown in figure 1. In Section 3 we analyze the comparative statics of the system. Section 4 contains several ideas for improvements and thereafter we conclude.

## 2 The system of equations

The special situation under consideration can be described as follows:

- There are two countries (A and B). Both countries produce with two factors, capital $K$ and labour $L$.
- Country A produces in Cone 1 and is specialized in producing goods $X$ and $Y$. Country B produces in Cone 2 and is specialized in producing goods $Y$ and $Z$.
- The production of good $X$ is more capital intensive compared to the others. Good $Z$ is produced labor intensively compared to the other two, while good $Y$ is in-between.
- There is no factor price equalization, thus $w_A \neq w_B$ and $r_A \neq r_B$.
- In each country, there is just one consumer who owns the factor endowments.
- Countries share the same technologies, thus the unit-cost functions are the same in both countries.

In a general equilibrium model with perfect competition, factor- and good markets have to be in equilibrium and the price of produced goods $(p_X, p_Y, p_Z)$ matches exactly the unit costs of firms $(c_X, c_Y, c_Z)$. The corresponding equilibrium conditions are as follows:

The zero profit conditions in production:

$$c_X(\omega_A) = r_A \cdot \frac{a_{KX}(\omega_A)}{A_{KX}} + w_A \cdot \frac{a_{LX}(\omega_A)}{A_{LX}} = p_X \tag{1}$$

$$c_Y(\omega_A) = r_A \cdot \frac{a_{KY}(\omega_A)}{A_{KY}} + w_A \cdot \frac{a_{LY}(\omega_A)}{A_{LY}} = p_Y \tag{2}$$

$$c_Y(\omega_B) = r_B \cdot \frac{b_{KY}(\omega_B)}{B_{KY}} + w_B \cdot \frac{b_{LY}(\omega_B)}{B_{LY}} = p_Y \tag{3}$$

$$c_Z(\omega_B) = r_B \cdot \frac{b_{KZ}(\omega_B)}{B_{KZ}} + w_B \cdot \frac{b_{LZ}(\omega_B)}{B_{LZ}} = p_Z \tag{4}$$

$\omega_A$ ($\omega_B$) represents factor prices in country A (B). The $a_{ij}$'s ($b_{ij}$'s) are the unit factor requirements of factor $i$ to produce good $j$ in country A (B). The unit factor requirements are functions of $\omega$, as profit-maximizing firms take the cost of input factors into account when they decide which amount of factors to employ to produce a good with a given technology.

At least in the traditional Solow-Growth-Model, technical change is the key driving force for long-term growth of nations. In our model, technological change enters the trading world through the parameters $A_{KX}, A_{LX}, ..., B_{KZ}, B_{LZ}$. A change in those parameters directly affects firms' production possibilities and – since we analyze a general equilibrium model – all endogenous variables. For example, a higher $A_{KX}$ means that less capital is needed to produce one unit of good X in country A. For the moment, think of them to have the value of unity. Then (1) to (4) are standard unit cost functions.

Full employment in factor markets:

$$K_A = X_A \cdot \frac{a_{KX}(\omega_A)}{A_{KX}} + Y_A \cdot \frac{a_{KY}(\omega_A)}{A_{KY}} \tag{5}$$

$$L_A = X_A \cdot \frac{a_{LX}(\omega_A)}{A_{LX}} + Y_A \cdot \frac{a_{LY}(\omega_A)}{A_{LY}} \tag{6}$$

$$K_B = Y_B \cdot \frac{b_{KY}(\omega_B)}{B_{KY}} + Z_B \cdot \frac{b_{KZ}(\omega_B)}{B_{KZ}} \tag{7}$$

$$L_B = Y_B \cdot \frac{b_{LY}(\omega_B)}{B_{LY}} + Z_B \cdot \frac{b_{LZ}(\omega_B)}{B_{LZ}} \tag{8}$$

The endowments of each country are represented by $K_{A,B}, L_{A,B}$. The conditions (5) to (8) state that factors are fully employed in the production of goods. Factors of country A are used to produce goods $X_A$ and $Y_A$, whereas factors of country B are used to produce goods $Y_B$ and $Z_B$. Factors are mobile between sectors and immobile internationally.

Equilibrium in good markets:

$$X_A = D_{XA}(\vec{p}; \underbrace{r_A K_A + w_A L_A}_{\text{income}}) + D_{XB}(\vec{p}; \underbrace{r_B K_B + w_B L_B}_{\text{income}}) \tag{9}$$

$$Y_A + Y_B = D_{YA}(\vec{p}; \underbrace{r_A K_A + w_A L_A}_{\text{income}}) + D_{YB}(\vec{p}; \underbrace{r_B K_B + w_B L_B}_{\text{income}}) \tag{10}$$

$$Z_B = D_{ZA}(\vec{p}; \underbrace{r_A K_A + w_A L_A}_{\text{income}}) + D_{ZB}(\vec{p}; \underbrace{r_B K_B + w_B L_B}_{\text{income}}) \tag{11}$$

Equations (9) to (11) equalize supply and demand in international good markets. Trade is free and frictionless.

In principle, we have now 11 equations for 11 unknown endogenous variables $(p_X, p_Y, p_Z, r_A, w_A, r_B, w_B, X_A, Y_A, Y_B, Z_B)$ and thus a determined system. *But*: In equations (1) to (11), a few endogenous variables appear as arguments of a unspecified function and thus cannot be isolated. For different values of $\omega_A$ or $\omega_B$, the parameters $a_{jk}(b_{jk})$ represent different amounts of factor-usage $j$ in country $A$ $(B)$ for the production of one unit of good $k$. These parameters depend on detailed properties of technology, most importantly on factor substitutability. Here, we treat the $a_{jk}$'s ($b_{jk}$'s) as parameters to the problem. In other words, we assume that the $a_{jk}(b_{jk})$ differ industry-per-industry and country-per-country for some unexplained reason. Later a shock will occur (technical change), but this will not affect the choice of the technical coefficients $a_{jk}(b_{jk})$. We limit the reaction of the general equilibrium to the shock to the case of a Leontief technology with no factor-substitutability at all.[2]

A similar problem (a few endogenous variables appear as arguments of a unspecified function) occurs in the demand-functions in equations (9) to (11). Demand (ordinary, Marshallian demand) depends in principle on prices and income. In order to isolate endogenous and exogenous variables, and parameters, we specify demand in more detail. We introduce the standard-assumption of homothetic preferences.

Homothetic preferences imply an expenditure function that is multiplicatively separable (Dixit / Norman 1980: 62f. and Appendix): $e(\vec{p}, u) = \psi(u) \cdot \overline{e}(\vec{p})$. The function $\psi(u)$ is an increasing function of $u$. As only ordinary utility matters, one can "take $\psi(u)$ itself as the indicator of utility, and then relabel it $u$." (DN80: 63) The expenditure function (here for good $X$) we specify is Cobb-Douglas:

$$e = u \cdot \prod_{i=1}^{N} \left(\frac{p_X}{\alpha_i}\right)^{1/3} =$$

$$u \cdot \left[\left(\frac{p_X}{1/3}\right)^{1/3} \cdot \left(\frac{p_Y}{1/3}\right)^{1/3} \cdot \left(\frac{p_Z}{1/3}\right)^{1/3}\right] = u \cdot \left(3 \cdot p_X^{\frac{1}{3}} \cdot p_Y^{\frac{1}{3}} \cdot p_Z^{\frac{1}{3}}\right) \tag{12}$$

From an expenditure function, (Hicksian) compensated demand functions $H(p, u)$ are derived by the corresponding partial derivatives with respect to the price of the good in question. The Hicksian demand for good $X$, for example, is:

$$\begin{aligned}
\frac{\partial e(\vec{p}, u)}{\partial p_X} = H_X(p, u) &= u \cdot 3 \cdot \tfrac{1}{3} \cdot p_X^{-\frac{2}{3}} \cdot p_Y^{\frac{1}{3}} \cdot p_Z^{\frac{1}{3}} \\
&= \frac{\frac{1}{3} \cdot 3 \cdot u \cdot p_X^{\frac{1}{3}} \cdot p_Y^{\frac{1}{3}} \cdot p_Z^{\frac{1}{3}}}{p_X} = \\
H_X &= \frac{\frac{1}{3} \cdot e}{p_X}
\end{aligned}$$

[2]In following research, this assumption should be relaxed, as the factor substitution might be ruled out for a "short-term" period, but in the long run, firms should be assumed to re-optimize their relative use of the different factors. One could think of technological progress to enter the world at the beginning of every period. At the beginning of the period, technology is Leontief and as time goes by firms re-decide about their new choice of technical coefficients. At the end of the period, a new "initial" general equilibrium would be "ready" for the next technological shock and so on.

Compensated demand functions $H(p,u)$ coincide with the ordinary demand functions $D(p,y)$ if the income $y$ is just right to attain the utility level in $H(p,u)$. The consistency condition $y = e(p,u)$ must hold. The relationship between $H(p,u)$ and $D(p,y)$ is (Dixit / Norman 1980: 60, 62):

$$H(p,u) = D(\vec{p}, y) = D(\vec{p}, e[\vec{p}, u]) \tag{13}$$

The step from Hicksian to Marshallian demand is now short: Given that income $y$ in the Marshallian demand $D_X(p,y)$ is just enough to reach the utility level $u$ in the Hicksian demand $H_X(p,u)$, then the two demands coincide. Then, we have as ordinary, Marshallian demand (for good X in country A) in the case of Cobb-Douglas-preferences:

$$D_{XA} = \frac{\frac{1}{3} \cdot y}{p_X} = \frac{\frac{1}{3} \cdot [(r_A K_A + w_A L_A)]}{p_X} \tag{14}$$

This specification of the demand functions will be used in the equilibrium conditions for the good markets. They have the standard properties of demand functions that are derived from identical and homothetic preferences. The amount of goods that are demanded by country A depends on prices and on the income level. The *composition* of the demand of country B is the same, nevertheless, the *levels* might be different if the income is different.

Having discussed our assumption on technical coefficients $a_{jk}$ ($b_{jk}$) and on demand, we can rewrite the simplified system of equations as follows:

$$c_X = r_A \cdot \frac{a_{KX}}{A_{KX}} + w_A \cdot \frac{a_{LX}}{A_{LX}} = p_A \tag{15}$$

$$c_Y = r_A \cdot \frac{a_{KY}}{A_{KY}} + w_A \cdot \frac{a_{LY}}{A_{LY}} = p_Y \tag{16}$$

$$c_Y = r_B \cdot \frac{b_{KY}}{B_{KY}} + w_B \cdot \frac{b_{LY}}{B_{LY}} = p_Y \tag{17}$$

$$c_Z = r_B \cdot \frac{b_{KZ}}{B_{KZ}} + w_B \cdot \frac{b_{LZ}}{B_{LZ}} = p_Z \tag{18}$$

$$K_A = X_A \cdot \frac{a_{KX}}{A_{KX}} + Y_A \cdot \frac{a_{KY}}{A_{KY}} \tag{19}$$

$$L_A = X_A \cdot \frac{a_{LX}}{A_{LX}} + Y_A \cdot \frac{a_{LY}}{A_{LY}} \tag{20}$$

$$K_B = Y_B \cdot \frac{b_{KY}}{B_{KY}} + Z_B \cdot \frac{b_{KZ}}{B_{KZ}} \tag{21}$$

$$L_B = Y_B \cdot \frac{b_{LY}}{B_{LY}} + Z_B \cdot \frac{b_{LZ}}{B_{LZ}} \tag{22}$$

$$X_A = \frac{1}{3} \cdot [r_A K_A + w_A L_A + r_B K_B + w_B L_B] \cdot p_X^{-1} \tag{23}$$

$$Y_A + Y_B = \frac{1}{3} \cdot [r_A K_A + w_A L_A + r_B K_B + w_B L_B] \cdot p_Y^{-1} \tag{24}$$

$$Z_B = \frac{1}{3} \cdot [r_A K_A + w_A L_A + r_B K_B + w_B L_B] \cdot p_Z^{-1} \tag{25}$$

This is now the complete system of equations that describes the general equilibrium in the "initial situation". The exogenous variables determine the endogenous variables. For our purpose of comparative statics the fully solved system of linear equations for levels is of less importance and the hilarious expressions are dispelled to Appendix A.1.[3]

# 3 Comparative Statics

We compute the total differentials for all equations. With total differentials at hand, we are then able to analyze any shock on the endogenous variables. In oder to improve readability, we again dispel our system of equations, this time in total differentials and matrix notation, to Appendix (A.2).

Solving the system of the eleven total differentials for the change of the endogenous variables is in general possible. However, the resulting expressions are extremely complicated and very long. For simplification and in order to ease the interpretation of the results, we decided to play with numbers. The first step is to find values for endowments and unit input requirements of the two countries that represent the following pattern:

- Country A is rich in capital relatively to country B
- The production of good $X$ in country A is the most capital intensive whereas the production of good $Z$ in country B is the most labor intensive. This sequence holds: $\frac{a_{KX}}{a_{LX}} > \frac{a_{KY}}{a_{LY}} > \frac{b_{KY}}{b_{LY}} > \frac{b_{KZ}}{b_{LZ}}$.
- The resulting values for the endogenous variables have to make sense economically, e.g. positive values for the prices and quantities. In our case factor prices should differ as we do not assume factor price equalization across both countries but two different cones of diversification.

[3]On request, there is an EXCEL-spreadsheet for this linear system available from the authors. One can test the reaction of changing exogenous variables and parameters on good and factor prices, and on quantities.

Unit factor requirements and endowments are chosen as follows:

$$\begin{aligned} a_{LX} &= 10 & a_{KX} &= 70 & a_{LY} &= 33 & a_{KY} &= 40 \\ b_{LY} &= 43 & b_{KY} &= 22 & b_{LZ} &= 50 & b_{KZ} &= 15 \\ K_A &= 115000 & L_A &= 50000 & K_B &= 35000 & L_B &= 110000 \end{aligned}$$

The corresponding values for the endogenous variables are:

$$\begin{aligned} w_A &= .0044 & r_A &= .0214 & w_B &= .0019 \\ r_B &= .0417 & X_A &= 940 & Y_A &= 1230 \\ Y_B &= 220 & Z_B &= 2011 & p_X &= 1.5431 \\ p_Z &= .7211 \end{aligned}$$

The total differentials for the system of equations – having used the above values – are displayed in equation (26) to (37):

$$d\left(p_X\right) = \frac{10\,d\left(w_A\right)}{A_{LX}} + \frac{70\,d\left(r_A\right)}{A_{KX}} - \frac{10\,w_A\,d\left(A_{LX}\right)}{A_{LX}^2} - \frac{70\,r_A\,d\left(A_{KX}\right)}{A_{KX}^2} \tag{26}$$

$$0 = \frac{33\,d\left(w_A\right)}{A_{LY}} + \frac{40\,d\left(r_A\right)}{A_{KY}} - \frac{33\,w_A\,d\left(A_{LY}\right)}{A_{LY}^2} - \frac{40\,r_A\,d\left(A_{KY}\right)}{A_{KY}^2} \tag{27}$$

$$0 = \frac{43\,d\left(w_B\right)}{B_{LY}} + \frac{22\,d\left(r_B\right)}{B_{KY}} - \frac{43\,w_B\,d\left(B_{LY}\right)}{B_{LY}^2} - \frac{22\,r_B\,d\left(B_{KY}\right)}{B_{KY}^2} \tag{28}$$

$$d\left(p_Z\right) = \frac{50\,d\left(w_B\right)}{B_{LZ}} + \frac{15\,d\left(r_B\right)}{B_{KZ}} - \frac{50\,w_B\,d\left(B_{LZ}\right)}{B_{LZ}^2} - \frac{15\,r_B\,d\left(B_{KZ}\right)}{B_{KZ}^2} \tag{29}$$

The simplification of setting $p_Y$ as numeraire has the result that the change of the price level of good $Y$ is zero. Contrary, the change in the other price levels can be either positive or negative depending on the exogenous shock.

Assuming fixed endowments in both countries, the change in the factor endowments is zero. The new equations for the factor markets (30) to (33) are stated below:

$$0 = \frac{40\, d\,(Y_A)}{A_{KY}} + \frac{70\, d\,(X_A)}{A_{KX}} - \frac{40\, Y_A\, d\,(A_{KY})}{A_{KY}^2} - \frac{70\, X_A\, d\,(A_{KX})}{A_{KX}^2} \tag{30}$$

$$0 = \frac{33\, d\,(Y_A)}{A_{LY}} + \frac{10\, d\,(X_A)}{A_{LX}} - \frac{33\, Y_A\, d\,(A_{LY})}{A_{LY}^2} - \frac{10\, X_A\, d\,(A_{LX})}{A_{LX}^2} \tag{31}$$

$$0 = \frac{15\, d\,(Z_B)}{B_{KZ}} + \frac{22\, d\,(Y_B)}{B_{KY}} - \frac{15\, Z_B\, d\,(B_{KZ})}{B_{KZ}^2} - \frac{22\, Y_B\, d\,(B_{KY})}{B_{KY}^2} \tag{32}$$

$$0 = \frac{50\, d\,(Z_B)}{B_{LZ}} + \frac{43\, d\,(Y_B)}{B_{LY}} - \frac{50\, Z_B\, d\,(B_{LZ})}{B_{LZ}^2} - \frac{43\, Y_B\, d\,(B_{LY})}{B_{LY}^2} \tag{33}$$

For the good markets, the total differentials are:

$$\begin{aligned} d\,(X_A) = {} & \frac{110000\, d\,(w_B)}{3\, p_X} + \frac{50000\, d\,(w_A)}{3\, p_X} + \frac{35000\, d\,(r_B)}{3\, p_X} + \frac{115000\, d\,(r_A)}{3\, p_X} \\ & + \frac{\left(-110000\, w_B - 50000\, w_A - 35000\, r_B - 115000\, r_A\right)\, d\,(p_X)}{3\, p_X^2} \end{aligned} \tag{34}$$

$$\begin{aligned} & d\,(Y_B) + d\,(Y_A) = \\ & \frac{110000\, d\,(w_B)}{3} + \frac{50000\, d\,(w_A)}{3} + \frac{35000\, d\,(r_B)}{3} + \frac{115000\, d\,(r_A)}{3} \end{aligned} \tag{35}$$

$$\begin{aligned} d\,(Z_B) = {} & \frac{110000\, d\,(w_B)}{3\, p_Z} \\ & + \frac{50000\, d\,(w_A)}{3\, p_Z} + \frac{35000\, d\,(r_B)}{3\, p_Z} + \frac{115000\, d\,(r_A)}{3\, p_Z} \\ & + \frac{\left(-110000\, w_B - 50000\, w_A - 35000\, r_B - 115000\, r_A\right)\, d\,(p_Z)}{3\, p_Z^2} \end{aligned} \tag{36}$$

Equations (26) to (37) describe an initial situation for two countries with two cones of diversification. For simplicity, we assume that in the initial situation the values for the technical coefficients $A_{KX}, ..., B_{LZ}$ are equal to one. The underlying reason for this assumption is that no bias in technology should be exist in the initial situation. Of course, technical change can be directed towards capital or labor or both in any combination. Biased technical change means that improved efficiency via technical change differs between factors (and maybe also between sectors).

To show how the endogenous variables, most importantly the factor prices, react to an exogenous shock – a change in technology – we perform an exercise in comparative statics. We solve the complete system of totally differentiated equations for the endogenous variables.

$$\begin{aligned} d(w_A) = .0033d(B_{LZ}) + .0014d(B_{LY}) - .0033d(B_{KZ}) - \\ .0023d(B_{KY}) + .1336d(A_{LY}) - .0243d(A_{LX}) + \\ 0.115d(A_{KY}) - .1109d(A_{KX}) \end{aligned} \tag{37}$$

$$\begin{aligned} d(r_A) = -.0027d(B_{LZ}) - .0011d(B_{LY}) + .0027d(B_{KZ}) + \\ .0019d(B_{KY}) - .0175d(A_{LY}) + .0201d(A_{LX}) - \\ .0174d(A_{KY}) + .0915d(A_{KX}) \end{aligned} \tag{38}$$

$$\begin{aligned} d(w_B) = -.0461d(B_{LZ}) - .1059d(B_{LY}) + .0461d(B_{KZ}) - \\ .0507d(B_{KY}) - .0944d(A_{LY}) - .0202d(A_{LX}) - \\ .0933d(A_{KY}) - .1257d(A_{KX}) \end{aligned} \tag{39}$$

$$\begin{aligned} d(r_B) = .0902d(B_{LZ}) + .3099d(B_{LY}) - .0902d(B_{KZ}) + \\ .1985d(B_{KY}) + .1846d(A_{LY}) + .0396d(A_{LX}) + \\ .1824d(A_{KY}) + .2456d(A_{KX}) \end{aligned} \tag{40}$$

$$\begin{aligned} d(X_A) = -141.4d(A_{LY}) - 47.6d(A_{LX}) + 141.4d(A_{KY}) + \\ 274.9d(A_{KX}) \end{aligned} \tag{41}$$

$$\begin{aligned} d(Y_A) = 247.4d(A_{LY}) + 83.29d(A_{LX}) - 42.84d(A_{KY}) - \\ 83.29d(A_{KX}) \end{aligned} \tag{42}$$

$$\begin{aligned} d(Y_B) = -688.3d(B_{LZ}) - 288.2d(B_{LY}) + 688.3d(B_{KZ}) + \\ 491.5d(B_{KY}) \end{aligned} \tag{43}$$

$$\begin{aligned} d(Z_B) = 1010.d(B_{LZ}) + 422.7d(B_{LY}) - 592.0d(B_{KZ}) - \\ 422.7d(B_{KY}) \end{aligned} \tag{44}$$

$$\begin{aligned} d(p_X) = -.1557d(B_{LZ}) - .0652d(B_{LY}) + .1557d(B_{KZ}) + \\ .1112d(B_{KY}) + .1133d(A_{LY}) + .0382d(A_{LX}) - \\ .0671d(A_{KY}) - .1304d(A_{KX}) \end{aligned} \tag{45}$$

$$\begin{aligned} d(p_Z) = -3.586d(B_{LZ}) - .6468d(B_{LY}) - .5371d(B_{KZ}) + \\ .4422d(B_{KY}) - 1.953d(A_{LY}) - .4188d(A_{LX}) - \\ 1.93d(A_{KY}) - 2.599d(A_{KX}) \end{aligned} \tag{46}$$

Several changes occur as a reason of any shock in technology. For example, the system of relative prices reacts to any shock. For the topic of factor price equalization, this is important. Figure 1 boils down the topic of factor price equalization to the position of unit value isoquants in the $K$-$L$ space. In the graphical analysis, the isoquants shift because of price movements. But they also move directly due to improved technical efficiency. In a graphical analysis, the two simultaneous movements of the isoquants are difficult to control. In a general equilibrium model, the simultaneous effects are captured and clearly displayed in the above stated equations.

Assuming improved technology only in the capital intensive sector of good $X$, and completely directed to capital ($A_{KX}$), the system would react with a decrease in both relative prices, $p_X$ and $p_Z$. The production of good $X$ ($Y$) in country $A$ increases (decreases), whereas the production in country $B$ is not affected. The production quantity in country $B$ cannot change because of the assumptions of fixed endowments and fixed unit input requirements. In country $B$, changes can only occur on the consumption side.

The rental rate of capital in country $A$ increases due to the shock and the wage rate decreases. In country $B$ opposite changes in factor rewards occur. In the initial situation, the wage-rental ratio ($\frac{w}{r}$) in the capital-rich country $A$ was relatively high compared to labor-rich country $B$. Improved efficiency in the use of capital in country $A$ and sector $X$ lowers the wage-rental ratio in country $A$ but also lowers the ratio in country $B$. Whether both ratios approach each other depends on detailed properties of the initial situation.[4] In our case, a specific shock only on $A_{KX}$, with the implicit assumptions about the values of endowments and the fixed unit factor requirements, does not result in factor price equalization.

The effect of all the other shocks to the system is summarized in Table 1.

| | $dr_A$ | $dw_A$ | $dr_B$ | $dw_B$ | $dX_A$ | $dY_A$ | $dY_B$ | $dZ_B$ | $dp_X$ | $dp_Z$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $A_{KX}$ | + | - | + | - | + | - | | | - | - |
| $A_{LX}$ | + | - | + | - | - | + | | | + | - |
| $A_{KY}$ | - | + | + | - | + | - | | | - | - |
| $A_{LY}$ | - | + | + | - | - | + | | | + | - |
| $B_{KY}$ | + | - | + | - | | | + | - | + | + |
| $B_{LY}$ | - | + | + | - | | | - | + | - | - |
| $B_{KZ}$ | + | - | - | + | | | + | - | + | - |
| $B_{LZ}$ | - | + | + | - | | | - | + | - | - |

Table 1: Comparative statics

[4]We did not perform the exercise with other specifications (different numbers) of the initial situation.

# 4 A few ideas: generalization, numeraires and endogenous technical change

In this paper, our initial situation was set out by just assuming that we have two countries, two cones of diversification and a specific pattern of production. Also, we worked with numbers for the comparative statics exercise in order to simplify the analysis.

Several generalizations are straightforward. Of course one could try to solve the system of differentiated equations (given in Appendix A.2) without assuming numbers. But the results we found were extremely large expressions that were hard to interpret. Therefore, a possible idea is to reduce the number of variables in the system.

One way for this reduction is to introduce further numeraires. For capital and as well for labor one could pick the unit capital input requirement of any sector in any country as a numeraire for capital, and, accordingly, any unit labor input requirement as a numeraire for labor. For example, one could define $a_{KX}$ as numeraire for capital and measure all the capital quantities in relation to the chosen unit capital requirement. Labour could be measured in units of $b_{LZ}$. Or one could pick any countries endowment with a factor as a numeraire for all factor quantities.

Throughout the paper, we were silent about why the shocks to technology in the different countries, sectors and directions (to $K$ or $L$) occur. The rest of the paper contains an idea whether and why the technical change is more directed towards $K$ or $L$. The main idea is that firms decide on how to make use of technical improvements. This idea has its origins in the 1960s. The literature and its modernization is summarized in Acemoglu (2001).

Figure 2 shows how firms decide whether the technical change is more directed towards $K$ or $L$. The plotted function $\hat{A}_{Li} = \Phi(\hat{A}_{Ki})$ offers different ways to distribute technical change on capital and labor. On the axis are the rates of change of the technical coefficients, $\frac{dA_{KX}}{A_{KX}}, ..., \frac{dB_{KZ}}{B_{KZ}}, \frac{dA_{LX}}{A_{LX}}, ..., \frac{dB_{LZ}}{B_{LZ}}$. At point $K$ a firm decides to direct technical change completely towards the factor capital. At point $L$ a firms uses technical change solely for improving the efficiency of labor input.

The question arises why firms choose capital improving or labor improving technical change. The cost function of a firm producing good $i$ ($i = X, Y, Z$) is as follows:

$$c_i = \; r \cdot \frac{a_{Ki}(\frac{w}{r})}{A_{Ki}} + w \cdot \frac{a_{Li}(\frac{w}{r})}{A_{Li}} \tag{47}$$

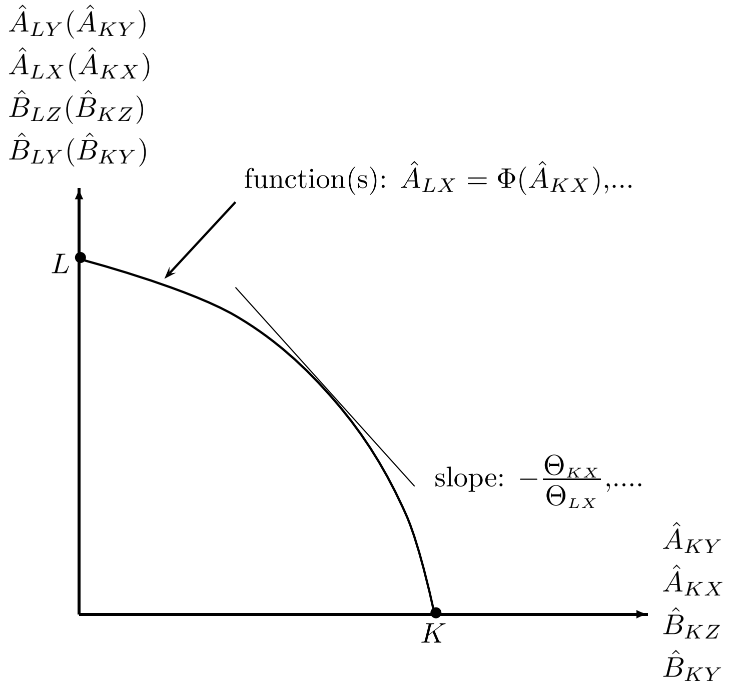


Figure 2: A firm's choice

Assume $w$ and $r$ as given. Totally differentiating and rearranging the unit cost function leads to the following result:

$$
\begin{aligned}
dc_i = & \quad \frac{\partial c_i}{\partial A_{Ki}} \cdot dA_{Ki} + \frac{\partial c_i}{\partial A_{Li}} \cdot dA_{Li} \\
= & -\frac{r \cdot a_{Ki} \cdot A_{Ki}}{A_{Ki}^2} \cdot \frac{dA_{Ki}}{A_{Ki}} - \frac{w \cdot a_{Li} \cdot A_{Li}}{A_{Li}^2} \cdot \frac{dA_{Li}}{A_{Li}} \\
= & -\frac{r \cdot a_{Ki}}{A_{Ki}} \cdot \frac{dA_{Ki}}{A_{Ki}} - \frac{w \cdot a_{Li}}{A_{Li}} \cdot \frac{dA_{Li}}{A_{Li}} \\
\Leftrightarrow \frac{dc_i}{c} = & \quad \frac{r \cdot (a_{Ki}/A_{Ki})}{c} \cdot \hat{A}_{Ki} + \frac{w \cdot (a_{Li}/A_{Li})}{c} \cdot \hat{A}_{Li}
\end{aligned}
\tag{48}
$$

Defining $\Theta_{Ki} \equiv \frac{r \cdot (a_{Ki}/A_{Ki})}{c_i}$ and $\Theta_{Li} \equiv \frac{w \cdot (a_{Li}/A_{Li})}{c_i}$ we can rewrite (48) to:

$$
\hat{c}_i = -[\Theta_{Ki} \cdot \hat{A}_{Ki} + \Theta_{Li} \cdot \hat{A}_{Li}] \tag{49}
$$

Equation (49) gives the change of unit costs ($\hat{c}_i$) resulting from technical change, given $w$ and $r$. $\Theta_{Ki}$ [$\Theta_{Li}$] is the cost of using "effective capital [labor]" in the production of good $i$, as a ratio to total costs of producing one unit of good $i$.

Now we are able to show why a firm chooses to direct technical change more in the direction of labor or capital. Any profit-maximizing firm wants to minimize costs, and thus to achieve a value for $\hat{c}_i$ that is as negative as possible, by choosing $\hat{A}_{Ki}$. With a choice of $\hat{A}_{Ki}$, a choice of $\hat{A}_{Li}$ is also made, via the function $\hat{A}_{Li} = \Phi(\hat{A}_{Ki})$. For a given $(w/r)$ and thus for given $a_{Ki}(w/r)$ and $a_{Li}(w/r)$, the first order condition for this problem is:

$$\begin{aligned} \frac{d\hat{c}_i}{d\hat{A}_{Ki}} = & \quad -[\Theta_{Ki} + \Theta_{Li} \cdot \frac{d\hat{A}_{Li}}{d\hat{A}_{Ki}}] = \quad -[\Theta_{Ki} + \Theta_{Li} \cdot \Phi'] \stackrel{!}{=} 0 \\ \Leftrightarrow \Phi' = & \qquad -\frac{\Theta_{Ki}}{\Theta_{Li}} \end{aligned} \tag{50}$$

$\Phi'$ is the slope of the function $\hat{A}_{Li} = \Phi(\hat{A}_{Ki})$ and is equal to the firm specific measure of capital intensity in production, $[-\ (\Theta_{Ki}/\Theta_{Li})]$. By this first order condition, one knows that for a capital intensive firm, with a relatively high (negative) value of $[-\ \Theta_{Ki}/\Theta_{Li}]$, the slope of its tangent to the function $\hat{B}_i = \Phi(\hat{A})$ is relatively steep. Thus: *A firm that is more capital intensive in its production implements technical change by choosing a high value of $\hat{A}_{Ki}$ relative to $\hat{A}_{Li}$.*[5] More simple: A capital-intensive firm uses technical change to improve the use of capital.

In the general equilibrium analysis, the idea of endogenous technical change can be used to replace the *change* of the technical coefficients $A_{Ki}, A_{Li}, B_{Ki}, B_{Li}$ by the function $\hat{A}_{Li} = \Phi(\hat{A}_{Ki})$. Depending on capital intensity, an overall technical change is directed towards capital and labor. Throughout the paper, the direction of technical change has been assumed to be exogenous to the general equilibrium. In this section, firms' decisions are modeled and depend on the individual firms capital intensity and the initial levels of factor and good prices.[6] We leave the implementation of this idea as a next step in understanding the mechanics of technical change in a general equilibrium trade model. Of course, we would prefer an explicit solution for the evolution of the general equilibrium to a comparative static exercise.

[5] 100 percent exact is: A firm that pays relatively much for capital (per unit of production) compared to labour chooses a high $\hat{A}_{Ki}$.

[6] In the here presented version of endogenous technical change replaces assumed knowledge about the size and pattern of technological shocks. But still, the specification of function $\hat{A}_{Li} = \Phi(\hat{A}_{Ki})$ has to be assumed as god-given.

# 5 Conclusion

As strongly advocated by Dixit / Norman (1980), this paper relies on general equilibrium theory for the analysis of a factor endowments model of trade. The systematic derivation and simplification of equilibrium conditions is demonstrated. Our workaround for the problem of too complicated expressions is to introduce specific values for parameters and exogenous variables. With this specific simplification we are on the one hand able to obtain easily interpretable comparative static effects, while on the other hand our results are heavily constrained and should be seen as an (counter-)example. Nevertheless, the ideas in section 4 may lead to a more general and less restricted version of an higher dimensional trade model with technical change.

# 6 References

Many of the calculations were done with the Computer Algebra System Maxima (version 5.9.0, http://maxima.sourcefourge.net). It is for free and available for several operating system (Mac OS X, Windows, UNIX/LINUX,....).

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# A Appendix

## A.1 Solutions for the endogenous variables

In order to solve the system described by equations (15) to (25), we define good $Y$ as the numeraire and set its price equal to one ($p_Y = 1$). By Walras' Law, we can now drop one of the equations. We chose (25) as it looks too complicated.

The fully solved system of linear equations in levels:[7]

$$w_A = -\frac{\begin{array}{r}\left(a_{KX}\,a_{KY}\,a_{LY} - a_{KY}^2\,a_{LX}\right)\,b_{KZ}\,L_B \\ +\left(2\,a_{KX}\,a_{KY}\,b_{KZ}\,b_{LY} - 2\,a_{KX}\,a_{KY}\,b_{KY}\,b_{LZ}\right)\,L_A \\ +\left(a_{KY}^2\,a_{LX} - a_{KX}\,a_{KY}\,a_{LY}\right)\,b_{LZ}\,K_B \\ +\left(\left(a_{KX}\,a_{LY} + a_{KY}\,a_{LX}\right)\,b_{KY}\,b_{LZ} + \left(-a_{KX}\,a_{LY} - a_{KY}\,a_{LX}\right)\,b_{KZ}\,b_{LY}\right)\,K_A\end{array}}{\begin{array}{r}\left(\left(a_{KX}\,a_{KY}\,a_{LY} - a_{KY}^2\,a_{LX}\right)\,b_{KY}\,b_{LZ} + \left(a_{KY}^2\,a_{LX} - a_{KX}\,a_{KY}\,a_{LY}\right)\,b_{KZ}\,b_{LY}\right)\,L_A \\ +\left(\left(a_{KY}\,a_{LX}\,a_{LY} - a_{KX}\,a_{LY}^2\right)\,b_{KY}\,b_{LZ} + \left(a_{KX}\,a_{LY}^2 - a_{KY}\,a_{LX}\,a_{LY}\right)\,b_{KZ}\,b_{LY}\right)\,K_A\end{array}}$$

$$r_A = \frac{\begin{array}{r}\left(a_{KX}\,a_{LY}^2 - a_{KY}\,a_{LX}\,a_{LY}\right)\,b_{KZ}\,L_B \\ +\left(\left(-a_{KX}\,a_{LY} - a_{KY}\,a_{LX}\right)\,b_{KY}\,b_{LZ} + \left(a_{KX}\,a_{LY} + a_{KY}\,a_{LX}\right)\,b_{KZ}\,b_{LY}\right)\,L_A \\ +\left(a_{KY}\,a_{LX}\,a_{LY} - a_{KX}\,a_{LY}^2\right)\,b_{LZ}\,K_B \\ +\left(2\,a_{LX}\,a_{LY}\,b_{KY}\,b_{LZ} - 2\,a_{LX}\,a_{LY}\,b_{KZ}\,b_{LY}\right)\,K_A\end{array}}{\begin{array}{r}\left(\left(a_{KX}\,a_{KY}\,a_{LY} - a_{KY}^2\,a_{LX}\right)\,b_{KY}\,b_{LZ} + \left(a_{KY}^2\,a_{LX} - a_{KX}\,a_{KY}\,a_{LY}\right)\,b_{KZ}\,b_{LY}\right)\,L_A \\ +\left(\left(a_{KY}\,a_{LX}\,a_{LY} - a_{KX}\,a_{LY}^2\right)\,b_{KY}\,b_{LZ} + \left(a_{KX}\,a_{LY}^2 - a_{KY}\,a_{LX}\,a_{LY}\right)\,b_{KZ}\,b_{LY}\right)\,K_A\end{array}}$$

[7] The command in MAXIMA is:
`S :: ALGSYS([D2, D3, D4, D5, D6, D7, D8, D9, D10, D11], [w_A, r_A, w_B, r_B, X_A, Y_A, Y_B, Z_B, p_X, p_Z]);`

$$w_B = -\frac{\begin{array}{r}(2\,a_{KX}\,a_{LY} - 2\,a_{KY}\,a_{LX})\,b_{KY}\,b_{KZ}\,L_B \\ + \left(a_{KX}\,b_{KY}\,b_{KZ}\,b_{LY} - a_{KX}\,b_{KY}^2\,b_{LZ}\right)\,L_A \\ + \left((a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\,b_{KY}\,b_{LZ} + (a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\,b_{KZ}\,b_{LY}\right)\,K_B \\ + \left(a_{LX}\,b_{KY}^2\,b_{LZ} - a_{LX}\,b_{KY}\,b_{KZ}\,b_{LY}\right)\,K_A\end{array}}{\begin{array}{r}\left((a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\,b_{KY}^2\,b_{LZ} + (a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\,b_{KY}\,b_{KZ}\,b_{LY}\right)\,L_B \\ + \left((a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\,b_{KY}\,b_{LY}\,b_{LZ} + (a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\,b_{KZ}\,b_{LY}^2\right)\,K_B\end{array}}$$

$$r_B = \frac{\begin{array}{r}\left((a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\,b_{KY}\,b_{LZ} + (a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\,b_{KZ}\,b_{LY}\right)\,L_B \\ + \left(a_{KX}\,b_{KZ}\,b_{LY}^2 - a_{KX}\,b_{KY}\,b_{LY}\,b_{LZ}\right)\,L_A \\ + (2\,a_{KY}\,a_{LX} - 2\,a_{KX}\,a_{LY})\,b_{LY}\,b_{LZ}\,K_B \\ + \left(a_{LX}\,b_{KY}\,b_{LY}\,b_{LZ} - a_{LX}\,b_{KZ}\,b_{LY}^2\right)\,K_A\end{array}}{\begin{array}{r}\left((a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\,b_{KY}^2\,b_{LZ} + (a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\,b_{KY}\,b_{KZ}\,b_{LY}\right)\,L_B \\ + \left((a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\,b_{KY}\,b_{LY}\,b_{LZ} + (a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\,b_{KZ}\,b_{LY}^2\right)\,K_B\end{array}}$$

$$X_A = -\frac{a_{KY}\,L_A - a_{LY}\,K_A}{a_{KX}\,a_{LY} - a_{KY}\,a_{LX}}$$

$$Y_A = \frac{a_{KX}\,L_A - a_{LX}\,K_A}{a_{KX}\,a_{LY} - a_{KY}\,a_{LX}}$$

$$Y_B = -\frac{b_{KZ}\,L_B - b_{LZ}\,K_B}{b_{KY}\,b_{LZ} - b_{KZ}\,b_{LY}}$$

$$Z_B = \frac{b_{KY}\,L_B - b_{LY}\,K_B}{b_{KY}\,b_{LZ} - b_{KZ}\,b_{LY}}$$

$$p_X = \frac{\begin{array}{r}(a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\; b_{KZ}\,L_B \\ +\,(a_{KX}\,b_{KZ}\,b_{LY} - a_{KX}\,b_{KY}\,b_{LZ})\; L_A \\ +\,(a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\; b_{LZ}\,K_B \\ +\,(a_{LX}\,b_{KY}\,b_{LZ} - a_{LX}\,b_{KZ}\,b_{LY})\; K_A\end{array}}{\begin{array}{r}(a_{KY}\,b_{KY}\,b_{LZ} - a_{KY}\,b_{KZ}\,b_{LY})\; L_A \\ +\,(a_{LY}\,b_{KZ}\,b_{LY} - a_{LY}\,b_{KY}\,b_{LZ})\; K_A\end{array}}$$

$$p_Z = -\frac{\begin{array}{r}(a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\; b_{KZ}\,L_B \\ +\,(a_{KX}\,b_{KZ}\,b_{LY} - a_{KX}\,b_{KY}\,b_{LZ})\; L_A \\ +\,(a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\; b_{LZ}\,K_B \\ +\,(a_{LX}\,b_{KY}\,b_{LZ} - a_{LX}\,b_{KZ}\,b_{LY})\; K_A\end{array}}{\begin{array}{r}(a_{KX}\,a_{LY} - a_{KY}\,a_{LX})\; b_{KY}\,L_B \\ +\,(a_{KY}\,a_{LX} - a_{KX}\,a_{LY})\; b_{LY}\,K_B\end{array}}$$

## A.2 Matrices for the differentiated system of equations

$$T \cdot U = V$$

$$T \cdot \begin{pmatrix} dp_X \\ dp_Y \\ dp_Z \\ dX_A \\ dY_A \\ dY_B \\ dZ_B \\ dr_A \\ dw_A \\ dr_B \\ dw_B \end{pmatrix} = \begin{pmatrix} -dA_{KX} \cdot a_{KX} \cdot r_A \cdot (A_{KX})^{-2} - dA_{LX} \cdot a_{LX} \cdot w_A \cdot (A_{LX})^{-2} \\ -dA_{KY} \cdot a_{KY} \cdot r_A \cdot (A_{KY})^{-2} - dA_{LY} \cdot a_{LY} \cdot w_A \cdot (A_{LY})^{-2} \\ -dB_{KY} \cdot b_{KY} \cdot r_B \cdot (B_{KY})^{-2} - dB_{LY} \cdot b_{LY} \cdot w_B \cdot (B_{LY})^{-2} \\ -dB_{KZ} \cdot b_{KZ} \cdot r_B \cdot (B_{KZ})^{-2} - dB_{LZ} \cdot b_{LZ} \cdot w_B \cdot (B_{LZ})^{-2} \\ dK_A + dA_{KX} \cdot X_A \cdot a_{KX} \cdot (A_{KX})^{-2} + dA_{KY} \cdot Y_A \cdot a_{KY} \cdot (A_{KY})^{-2} \\ dL_A + dA_{LX} \cdot X_A \cdot a_{LX} \cdot (A_{LX})^{-2} + dA_{LY} \cdot Y_A \cdot a_{LY} \cdot (A_{LY})^{-2} \\ dK_B + dB_{KY} \cdot Y_B \cdot b_{KY} \cdot (B_{KY})^{-2} + dB_{KZ} \cdot Z_B \cdot b_{KZ} \cdot (B_{KZ})^{-2} \\ dL_B + dB_{LY} \cdot Y_B \cdot b_{LY} \cdot (B_{LY})^{-2} + dB_{LZ} \cdot Z_B \cdot b_{LZ} \cdot (B_{LZ})^{-2} \\ \frac{1}{3} \cdot p_X^{-1} \cdot (dK_A \cdot r_A + dL_A \cdot w_A + dK_B \cdot r_B + dL_B \cdot w_B) \\ \frac{1}{3} \cdot p_Y^{-1} \cdot (dK_A \cdot r_A + dL_A \cdot w_A + dK_B \cdot r_B + dL_B \cdot w_B) \\ \frac{1}{3} \cdot p_Z^{-1} \cdot (dK_A \cdot r_A + dL_A \cdot w_A + dK_B \cdot r_B + dL_B \cdot w_B) \end{pmatrix}$$

with $T =$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_{KX}}{A_{KX}} & -\frac{a_{LX}}{A_{LX}} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{a_{KY}}{A_{KY}} & -\frac{a_{LY}}{A_{LY}} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{b_{KY}}{B_{KY}} & -\frac{b_{LY}}{B_{LY}} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{b_{KZ}}{B_{KZ}} & -\frac{b_{LZ}}{B_{LZ}} \\
0 & 0 & 0 & \frac{a_{KX}}{A_{KX}} & \frac{a_{KY}}{A_{KY}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{a_{LX}}{A_{LX}} & \frac{a_{LY}}{A_{LY}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{b_{KY}}{B_{KY}} & \frac{b_{KZ}}{B_{KZ}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{b_{LY}}{B_{LY}} & \frac{b_{LZ}}{B_{LZ}} & 0 & 0 & 0 & 0 \\
\frac{\frac{1}{3}\cdot y}{p_X^2} & 0 & 0 & 1 & 0 & 0 & 0 & \frac{-\frac{1}{3}\cdot K_A}{p_X} & \frac{-\frac{1}{3}\cdot L_A}{p_X} & \frac{-\frac{1}{3}\cdot K_B}{p_X} & \frac{-\frac{1}{3}\cdot L_B}{p_X} \\
0 & \frac{\frac{1}{3}\cdot y}{p_Y^2} & 0 & 0 & 1 & 1 & 0 & \frac{-\frac{1}{3}\cdot K_A}{p_Y} & \frac{-\frac{1}{3}\cdot L_A}{p_Y} & \frac{-\frac{1}{3}\cdot K_B}{p_Y} & \frac{-\frac{1}{3}\cdot L_B}{p_Y} \\
0 & 0 & \frac{\frac{1}{3}\cdot y}{p_Z^2} & 0 & 0 & 0 & 1 & \frac{-\frac{1}{3}\cdot K_A}{p_Z} & \frac{-\frac{1}{3}\cdot L_A}{p_Z} & \frac{-\frac{1}{3}\cdot K_B}{p_Z} & \frac{-\frac{1}{3}\cdot L_B}{p_Z}
\end{pmatrix}
$$

where $y = K_A \cdot r_A + L_A \cdot w_A + K_B \cdot r_B + L_B \cdot w_B$.